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Nearest Interval-Valued Approximation of Interval-Valued Fuzzy Numbers

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ABSTRACT

In this paper, we proposed a new interval-valued approximation of interval-valued fuzzy numbers, which is the best one with respect to a certain measure of distance between interval-valued fuzzy numbers. Also, a set of criteria for interval-valued approximation operators is suggested.

Keywords: Fuzzy number, Interval-valued approximation, distance.

1. Introduction

Fuzzy numbers play significant role among all fuzzy sets since the predominant carrier of information are numbers.

As a generalization of an ordinary Zadeh fuzzy set, the notion of interval-valued fuzzy set was suggested for the first time by Gorzalczany (2002) and

Turksen (1986). Also, to this field, Wang and Li (1998) defined interval-valued fuzzy numbers and gave their extended operations.

(Hong and Lee, 1986) continuous Wang's work and expressed some algebraic properties of interval-valued fuzzy numbers. Also, a metric on interval-valued fuzzy numbers were proposed and investigate some convergence theorems for sequences of interval-valued fuzzy numbers with respect to the metric are treated.

However, sometimes we have to approximate a given fuzzy set by a crisp one. If we then use a defuzzification operator which replaces a fuzzy set by a single number we generally loose too many important information. Therefore, a crisp set approximation of a fuzzy set is often advisable.

In this paper, we proposed a new interval-valued distance to approximate interval-valued fuzzy numbers. To this end, we prove the metric properties of our proposed distance. Similar Grzegorzewski (2002), we propose nearest interval-valued approximation of interval-valued fuzzy number with respect to proposed metric. Also, we extend the good-reasonable properties of interval-valued approximate operator and check their properties to our approximate operator.

Our paper is organized as follow: in Section2 we express some basic concepts and interval operations for interval-valued fuzzy numbers. in Section3 we proposed *Nearest interval-valued approximation* with respect new interval-valued metric. Additionally, we investigate some proposed criteria to present a good-reasonable interval-valued approximate operators.

2. Preliminaries

In this section, we gathered some needed definitions and theorems which will be applied in future sections Kaufmann and Gupta (1985) and Zimmermann (1996).

Also, we use the same notations used in Wang and Li (1998). Let $I = [0, 1]$ and let $[I] = \{[a, b]| \;\; a \leq b, \; a, b \in I\}$. For any $a \in I$, define $\overline{a} = [a, a]$.

Definition 2.1. *If $a_t \in I$, $t \in T$, then we define $\bigvee_{t \in T} a_t = \sup\{a_t : \;\; t \in T\}$*

and $\bigwedge_{t\in T} a_t = \inf\{a_t :\ t\in T\}$. *Also, is defined for* $[a_t, b_t] \in [I],\ t \in T$

$$\bigvee_{t\in T}[a_t,b_t] = [\bigvee_{t\in T} a_t, \bigvee_{t\in T} b_t]$$

$$\bigwedge_{t\in T}[a_t,b_t] = [\bigwedge_{t\in T} a_t, \bigwedge_{t\in T} b_t]$$

Definition 2.2. *Let* $[a_1, b_1]$,$[a_2, b_2] \in [I]$, *then*

$$[a_1,b_1] = [a_2,b_2] \quad iff \quad a_1 = a_2,\ b_1 = b_2$$

$$[a_1,b_1] \leq [a_2,b_2] \quad iff \quad a_1 \leq a_2,\ b_1 \leq b_2$$

$$[a_1,b_1] < [a_2,b_2] \quad iff \quad [a_1,b_1] \leq [a_2,b_2],\ but,\ a_1 \neq a_2,\ b_1 \neq b_2$$

Definition 2.3. *Let* X *be an ordinary nonempty set. Then the mapping* $A : X \longrightarrow [I]$ *is called an interval-valued fuzzy set on* X. *All interval-valued fuzzy sets on* X *are denoted by* $IF(X)$.

Definition 2.4. *If* $A \in IF(X)$, *let* $A(x)$=$[A^-(x)$,$A^+(x)]$, *where* $x \in X$. *Then two ordinary sets* $A^- : X \longrightarrow I$ *and* $A^+ : X \longrightarrow I$ *are called lower fuzzy set and upper fuzzy set about A, respectively.*

Definition 2.5. *Let* $A \in IF(X)$, $[r_1, r_2] \in [I]$. *Then we called*

$$A_{[r_1,r_2]} = \{x \in X|\ \ A^-(x) \geq r_1,\ A^+(x) \geq r_2\}$$

$$A_{(r_1,r_2)} = \{x \in X|\ \ A^-(x) > r_1,\ A^+(x) > r_2\}$$

the $[r_1, r_2]-$*level set and* $(r_1, r_2)-$*level set of* A *respectively. And let* $A^-_r = \{x \in X|\ A^-(x) \geq r\}$, $A^+_r = \{x \in X|\ A^+(x) \geq r\}$.

Definition 2.6. *Let* $A \in IF(\mathbb{R})$, *i.e.* $A : R \longrightarrow I$. *Assume the following conditions are satisfied:*
1- A is normal, i.e. there exists $x_0 \in R$ *such that* $A(x_0) = 1$,
2- For arbitrary $[r_1, r_2] \in [I] - \{\bar{0}\}$, $A_{[r_1,r_2]}$ *is closed bounded interval,*
then we call A an interval-valued fuzzy number on real line R.

Let $IF^*(\mathbb{R})$ denote the set of all interval-valued fuzzy numbers on $\mathbb{R}$, and write $[I]^+ = [I] - \{\bar{0}\}$.

Definition 2.7. *Let* $A \in IF(\mathbb{R})$. *The A is called an interval convex set, if for any* $x, y \in R$ *and* $\lambda \in [0, 1]$, *we have*

$$A(\lambda x + (1-\lambda)y) \geq A(x) \wedge A(y).$$

Definition 2.8. *Let* A,$B \in IF(\mathbb{R})$, $\bullet \in \{+,-,\cdot,\div\}$. *define their extended operations to* $(A \bullet B)(z) = \bigvee_{z=x\bullet y}(A(x) \wedge B(y))$. *For each* $[r_1, r_2] \in [I]^+$ *we write*

$$A[r_1, r_2] \bullet B[r_1, r_2] = \{x \bullet y| \ \ x \in A_{[r_1,r_2]},\ y \in B_{[r_1,r_2]}\}$$

Definition 2.9. *Let* $A\in IF^*(\mathbb{R})$, *Then* A *is called positive interval-valued fuzzy number, if* $A(x) = \bar{0}$ *whenever,* $x \leq 0$, *and* A *is called a negative interval-valued fuzzy number, if* $A(x) = \bar{0}$ *whenever,* $x \geq 0$.

All positive interval-valued fuzzy numbers and all negative interval-valued fuzzy numbers are denoted by $IF^*_+(\mathbb{R})$ and $IF^*_-(R)$, respectively.

Theorem 2.1. *Let* $A, B \in IF^*(\mathbb{R})$, $\bullet \in \{+,-,\cdot,\div\}$. *Then*

$$(A \bullet B)(z) = \left[(A^- \bullet B^-)(z),\ (A^+ \bullet B^+)(z)\right]$$

Corollary 2.1. *Let* $A, B \in IF^*(\mathbb{R})$, *for any* $[r_1, r_2] \in [I]^+$,

$$(A \bullet B)_{[r_1,r_2]} = (A_{[r_1,r_2]} \bullet B_{[r_1,r_2]})$$

where, $\bullet \in \{+,-,\cdot,\div\}$. *and* $B \in IF^*_+(\mathbb{R}) \cup IF^*_-(\mathbb{R})$ *whenever* $\bullet$ *chooses* $\div$.

Definition 2.10. *Let* $A\in IF^*(\mathbb{R})$, *then the power of* A, A^P, *where* $p \in \mathbb{R}$ *is denoted as*

$$A^p(x) = \left[\left(A^-(x)\right)^p, \left(A^+(x)\right)^p\right],$$

where,

$$\left(A^-(x)\right)^p = \left[\left(A_1^-(x)\right)^p, \left(A_2^-(x)\right)^p\right]$$

and

$$\left(A^+(x)\right)^p = \left[\left(A_1^+(x)\right)^p, \left(A_2^+(x)\right)^p\right]$$

Definition 2.11. *Let* A,$B\in IF^*(\mathbb{R})$, *then* $(A \bullet B)^p$, *where* $\bullet \in \{+,-,\cdot,\div\}$ *and* $p \in \mathbb{R}$ *is denoted as*

$$(A \bullet B)^p = \left[(A^-(x))^p \bullet (B^-(x))^p, (A^+(x))^p \bullet (B^+(x))^p\right]$$

In this paper, r-cuts of lower fuzzy number $A^-(x)$ is expressed as $[A_1^-(r), A_2^-(r)]$,where

$$A_1^-(r) = \inf\{x \in \mathbb{R} :\ A^-(x) \geq r\},\quad 0 \leq r \leq 1$$

$$A_2^-(r) = \sup\{x \in \mathbb{R} :\ A^-(x) \geq r\},\quad 0 \leq r \leq 1$$

and the r-cut of upper fuzzy number $A^+(x)$ is expressed as $[A_1^+(r), A_2^+(r)]$,where

$$A_1^+(r) = \inf\{x \in \mathbb{R} : \ A^+(x) \geq r\}, \quad 0 \leq r \leq 1$$

$$A_2^+(r) = \sup\{x \in \mathbb{R} : \ A^+(x) \geq r\}, \quad 0 \leq r \leq 1.$$

For two arbitrary interval-valued fuzzy numbers A with r-cut representation $[A_r^-, A_r^+]$, where $A_r^- = [A_1^-(r), A_2^-(r)]$ and $A_r^+ = [A_1^+(r), A_2^+(r)]$, interval-valued fuzzy number B with r-cut representation $[B_r^-, B_r^+]$, where $B_r^- = [B_1^-(r), B_2^-(r)]$ and $B_r^+ = [B_1^+(r), B_2^+(r)]$, the quantity

$$d_I[A, B] = \left[d_I^-, d_I^+\right] \tag{1}$$

where,

$$d_I^- = \left[\sqrt{\int_0^1 (A_1^-(r) - B_1^-(r))^2 + (A_2^-(r) - B_2^-(r))^2} \right],$$

$$d_I^+ = \left[\sqrt{\int_0^1 (A_1^+(r) - B_1^+(r))^2 + (A_2^+(r) - B_2^+(r))^2} \right],$$

is the distance between A and B.

3. Nearest Interval-Valued Approximation

Suppose that we want to approximate an interval-valued fuzzy number by an interval-valued interval. To this end, we have to use an interval operator $C : IF^*(\mathbb{R}) \longrightarrow IP^*(\mathbb{R})$ where, $IP^*(\mathbb{R})$ is family of closed interval-valued intervals. In this section, we will propose an interval-valued approximation operator called the *nearest interval-valued interval approximation.*

Suppose A is an interval-valued fuzzy number A and its $r-$cut as $[A_r^-, A_r^+]$. Given A we will try to find a closed interval-valued interval $C_{d_I}(A)$ which is the nearest to A with respect to metric d_I. We can do it since each interval-valued interval is also an interval-valued fuzzy number with constant $r-$cuts for all $r \in (0, 1]$. Hence, let $C_{d_I}(A) = [C_L, C_U]$, where

$$C_L = [C_L^1, C_L^2], \qquad C_U = [C_U^1, C_U^2].$$

Now, we have to minimize $d_I(A, C_{d_I}(A))$ with respect to C_L^1, C_L^2, C_U^1 *and* C_U^2. In order to minimize $d_I(A, C_{d_I}(A))$ it suffices to minimize function $D_I(A, C_{d_I}(A)) =$

$d_I^2(A, C_{d_I}(A))$. Thus we have to find partial derivatives as follow:

$$\frac{\partial D(C_L^1, C_L^2, C_U^1, C_U^2)}{\partial C_L^1} = \left[-2\int_0^1 A_1^-(r) - C_L^1(r), 0\right] = \bar{0}$$

$$\frac{\partial D(C_L^1, C_L^2, C_U^1, C_U^2)}{\partial C_L^2} = \left[0, -2\int_0^1 A_2^-(r) - C_L^2(r)\right] = \bar{0}$$

$$\frac{\partial D(C_L^1, C_L^2, C_U^1, C_U^2)}{\partial C_U^1} = \left[-2\int_0^1 A_1^+(r) - C_U^1(r), 0\right] = \bar{0}$$

$$\frac{\partial D(C_L^1, C_L^2, C_U^1, C_U^2)}{\partial C_U^2} = \left[0, -2\int_0^1 A_2^+(r) - C_U^2(r)\right] = \bar{0}$$

The solution is

$$C_L^1 = \int_0^1 A_1^-(r)dr, \quad C_L^2 = \int_0^1 A_2^-(r)dr$$

$$C_U^1 = \int_0^1 A_1^+(r)dr, \quad C_U^2 = \int_0^1 A_2^+(r)dr$$

Moreover, since

$$\det\left[\frac{\partial^4 D(C_L^1, C_L^2, C_U^1, C_U^2)}{\partial C_L^i \partial C_U^j}\right] = 16 > 0, \ i+j = 4, \ i, j = 0, \ldots, 4.$$

and $\frac{\partial^4 D(C_L^1, C_L^2, C_U^1, C_U^2)}{\partial C_L^4} > 0$, then C_L and C_U given by $C_L = [C_L^1, C_L^2]$ and $C_U = [C_U^1, C_U^2]$, actually minimize $D(C_L^1, C_L^2, C_U^1, C_U^2)$ and simultaneously, minimize $d_I(A, C_{d_I}(A))$. Therefore the interval-valued interval

$$C_{d_I}(A) = \left[[\int_0^1 A_1^-(r)dr, \int_0^1 A_2^-(r)dr], [\int_0^1 A_1^+(r)dr, \int_0^1 A_2^+(r)dr]\right] \quad (2)$$

is indeed the nearest interval-valued approximation of interval-valued fuzzy number A with respect to metric d_I.

3.1 Good-reasonable properties of interval-valued interval approximation

in this subsection, we proposed some good-reasonable properties of interval-valued approximation of interval-valued fuzzy numbers as criteria for approximation , similar works which is explained and discussed by (Grzegorzewski and Mrowka, 2005).

3.1.1 Translation Invariance:

We say that an interval-valued approximation operator C_{d_I} is *translation Invariance* if $C_{d_I}(A+\zeta) = C_{d_I}(A) + \zeta$, for all $\zeta \in \mathbb{R}$.

Proposition 3.1. *: Our proposed interval-valued approximation $C_{d_I}(.)$ is not translation invariance.*

Proof: Let $A \in IF^*(\mathbb{R}), \zeta \in \mathbb{R}$ then

$$C_{D_I}(A+\zeta) = \left[\int_0^1 \left((A(r)+\zeta)_1^-\right)^2 + \left((A(r)+\zeta)_2^-\right)^2, \int_0^1 \left((A(r)+\zeta)_1^+\right)^2 + (((A(r)+\zeta)_2^+)^2\right] =$$

$$\left[\int_0^1 \left(A_1^-(r)\right)^2 + \left(A_2^-(r)\right)^2, \int_0^1 \left(A_1^+(r)\right)^2 + \left(A_2^+(r)\right)^2\right] + 2\left[\zeta^2, \zeta^2\right] = C_{D_I} + 2\chi_{\zeta^2}$$

Thus we have

$$C_{D_I}(A+\zeta) = C_{D_I}(A) + 2\chi_{\zeta^2} \Longrightarrow C_{d_I}(A+\zeta) = C_{d_I}(A) + \sqrt{2}\chi_\zeta$$

3.1.2 Scale invariant:

We say that an interval-valued approximation operator C_{d_I} is *scale invariant* if $C_{d_I}(\lambda A) = \lambda C_{d_I}(A)$, for all $\lambda \in \mathbb{R} - \{0\}$.

Proposition 3.2. *: Our proposed interval-valued approximation $C_{d_I}(.)$ is scale invariant.*

Proof: Let $A \in IF^*(\mathbb{R})$, then

$$\begin{aligned} C_{D_I}(\lambda A) &= \left[\int_0^1 (\lambda A_1^-)^2 + (\lambda A_2^-)^2, \int_0^1 (\lambda A_1^+)^2 + (\lambda A_2^+)^2\right] \\ &= \lambda^2 \left[\int_0^1 (A_1^-)^2 + (A_2^-)^2, \int_0^1 (A_1^+)^2 + (A_2^+)^2\right] \\ &= \lambda^2 C_{D_I}(A) \end{aligned}$$

Thus, $C_{d_I}(\lambda A) = \lambda C_{d_I}(A)$.

3.1.3 Monotony:

The criterion of monotony states that for any two interval-valued fuzzy numbers A and B holds

$$if \quad A \subseteq B \quad then \quad C_{d_I}(A) \subseteq C_{d_I}(B).$$

Proposition 3.3. *: Our proposed interval-valued approximation $C_{d_I}(.)$ is monotone.*

proof: Let $A, B \in IF^*(\mathbb{R})$ and $A \subseteq B$ then there exists a function $\Theta(r) = [\Theta_1, \Theta_2] \geq \bar{0}$ and $\Psi(r) = [\Psi_1, \Psi_2] \geq \bar{0}$ such that

$$A^-(r) = B^-(r) + \Theta(r) \equiv A_1^-(r) = B_1^-(r) + \Theta_1(r), \;\; A_2^-(r) = B_2^-(r) + \Theta_2(r)$$

$$A^+(r) = B^+(r) - \Psi(r) \equiv A_1^+(r) = B_1^+(r) - \Psi_1(r), \;\; A_2^+(r) = B_2^+(r) - \Psi_2(r)$$

and by using definition distance $C_{d_I}(A)$ we have

$$C_{d_I}(A) = \left[[\int_0^1 A_1^-(r)dr, \int_0^1 A_2^-(r)dr], [\int_0^1 A_1^+(r)dr, \int_0^1 A_2^+(r)dr] \right]$$

then by substitution of equivalent values from (3) we have two inequalities as follow:

$$\int_0^1 [B_1^-(r)+\Theta_1(r)]^2 dr + \int_0^1 [B_2^-(r)+\Theta_2(r)]^2 dr \geq \int_0^1 [B_1^-(r)]^2 dr + \int_0^1 [B_2^-(r)]^2 dr,$$

$$\int_0^1 [B_1^+(r)-\Psi_1(r)]^2 dr + \int_0^1 [B_2^+(r)-\Psi_2(r)]^2 dr \leq \int_0^1 [B_1^+(r)]^2 dr + \int_0^1 [B_2^+(r)]^2 dr,$$

or equivalently, is obtained

$$[C_L]_A \geq [C_L]_B, \qquad [C_U]_A \leq [C_U]_B$$

Therefore, $C_{D_I}(A) \subseteq C_{D_I}(B)$ and should be $C_{d_I}(A) \subseteq C_{d_I}(B)$.

3.1.4 Continuity:

An interval-valued interval approximation operator $C_{d_I}(A)$ is called continuous if for any $A, B \in IF^*$ we have

$$\forall \bar{\epsilon}, \; \exists \bar{\delta} : \quad d_I[A, B] < \bar{\delta} \Longrightarrow \; d_I[C_{d_I}(A), C_{d_I}(B)] < \bar{\epsilon}$$

where, $d_I : IF^*(\mathbb{R}) \times IF^*(\mathbb{R}) \longrightarrow [\mathbb{R}^+, \mathbb{R}^+]$ denotes a metric defined on the family of all interval-valued fuzzy numbers.

Proposition 3.4. *: Our proposed interval-valued approximation $C_{d_I}(.)$ is continuous.*

Proof: Let $A, B \in IF^*(\mathbb{R})$ then
$D_I\left[C_{d_I}(A), C_{d_I}(B)\right] =$

$$\left[\int_0^1 \{(C_L^1)_A - (C_L^1)_B\}^2 + \{(C_L^2)_A - (C_L^2)_B\}^2, \int_0^1 \{(C_U^1)_A - (C_U^1)_B\}^2 + \{(C_U^2)_A - (C_U^2)_B\}^2\right] =$$

$$\left[\{(C_L^1)_A - (C_L^1)_B\}^2 + \{(C_L^2)_A - (C_L^2)_B\}^2, \{(C_U^1)_A - (C_U^1)_B\}^2 + \{(C_U^2)_A - (C_U^2)_B\}^2\right] =$$

$$\left[\left(\int_0^1 A_1^- - B_1^-\right)^2 + \left(\int_0^1 A_2^- - B_2^-\right)^2, \left(\int_0^1 A_1^+ - B_1^+\right)^2 + \left(\int_0^1 A_2^+ - B_2^+\right)^2\right] \leq$$

$$\left[\int_0^1 (A_1^- - B_1^-)^2 + \int_0^1 (A_2^- - B_2^-)^2, \int_0^1 (A_1^+ - B_1^+)^2 + \int_0^1 (A_2^+ - B_2^+)^2\right] = D_I[A, B]$$

or equivalently, $d_I\left[C_{d_I}(A), C_{d_I}(B)\right] \leq d_I[A, B]$.

3.2 Metric Properties

In this part, we prove that our proposed interval-valued distance d_I has metric properties.

Metric properties of d_I:

1- $d_I[A, B] \geq \bar{0}$,

2- $d_I[A, B] = d_I(B, A)$,

3- $d_I[A, C] \leq d_I[A, B] + d_I[B, C]$,

4- $d_I[A, B] = \bar{0} \Longrightarrow A = B$.

Let $A, B, C \in IF^*(\mathbb{R})$, then case(1) and case(2) by applying definition of d_I is obvious. Thus we proof case(3) and case(4).

proof case(3):

$D_I[A, C] =$

$$\left[\int_0^1 (A_1^- - C_1^-)^2 + (A_2^- - C_2^-)^2, \int_0^1 (A_1^+ - C_1^+)^2 + (A_2^+ - C_2^+)^2\right] =$$

$$[\int_0^1 (A_1^- - C_1^- + B_1^- - B_1^-)^2 + (A_2^- - C_2^- + B_2^- - B_2^-)^2,$$

$$\int_0^1 (A_1^+ - C_1^+ + B_1^+ - B_1^+)^2 + (A_2^+ - C_2^+ + B_2^+ - B_2^+)^2] =$$

$$[\int_0^1 (A_1^- - B_1^-)^2 + (B_1^- - C_1^-)^2 + (A_2^- - B_2^-)^2 + (B_2^- - C_2^-)^2,$$

$$\int_0^1 (A_1^+ - B_1^+)^2 + (B_1^+ - C_1^+)^2 + (A_2^+ - B_2^+)^2 + (B_2^+ - C_2^+)^2] = D_I[A,B] + D_I[B,C] \leq$$

$$D_I[A,B] + D_I[B,C] + 2d_I[A,B]d_I[B,C] = (D_I[A,B] + D_I[B,C])^2, \ thus,$$

$$d_I[A,C] \leq d_I[A,B] + d_I[B,C]$$

proof case(4):

Let $d_I[A,B] = \left[d_I^-, d_I^+\right] = \bar{0}$ then,

$$d_I^- = \left[\sqrt{\int_0^1 (A_1^-(r) - B_1^-(r))^2 + (A_2^-(r) - B_2^-(r))^2} \right] = 0,$$

$$d_I^+ = \left[\sqrt{\int_0^1 (A_1^+(r) - B_1^+(r))^2 + (A_2^+(r) - B_2^+(r))^2} \right] = 0,$$

Therefore, we have

$$\int_0^1 (A_1^-(r) - B_1^-(r))^2 + (A_2^-(r) - B_2^-(r))^2 = 0,$$

$$\int_0^1 (A_1^+(r) - B_1^+(r))^2 + (A_2^+(r) - B_2^+(r))^2 = 0.$$

Hence

$$(A_1^-(r) - B_1^-(r))^2 = 0, \quad (A_2^-(r) - B_2^-(r))^2 = 0$$
$$(A_1^+(r) - B_1^+(r))^2 = 0, \quad (A_2^+(r) - B_2^+(r))^2 = 0$$

So, should have

$$A_1^-(r) = B_1^-(r), \quad A_2^-(r) = B_2^-(r), \quad A_1^+(r) = B_1^+(r), \quad A_2^+(r) = B_2^+(r)$$

Therefore, $A = B$ and the proof is complete.
Also, our proposed interval-valued approximate operator is coincide with interval approximate operator where, fuzzy numbers are replaced with interval-valued fuzzy numbers. So, if

$$A_r = A_r^- = A_r^+, \quad B_r = B_r^- = B_r^+$$

then

$$d_I^- = d_I^+ = \sqrt{\int_0^1 (A_r^- - B_r^-)^2 + (A_r^+ - B_r^+)^2}$$

4. Conclusion

In this paper, we proposed a new interval-valued approximation of interval-valued fuzzy numbers, which is the best one with respect to a new measure of distance between interval-valued fuzzy numbers. Also, a set of criteria for interval-valued approximation operators is suggested. Also, we see that our proposed interval-valued distance is coincide with ordinary distance where fuzzy number are replaced with interval-valued fuzzy number. For future work we want to extend trapezoidal approximation to interval-valued fuzzy numbers.

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